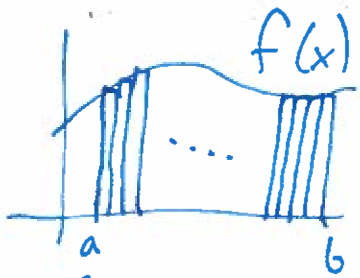
6.1: Volumes Using Cross Sections

$\boxed{\text{See Maplet on Volumes of Revolution.}}$

We wish to imitate Riemann Sums to Volumes.

2-dim



$$\sum_{i=1}^{n} f(x_i)\Delta x$$

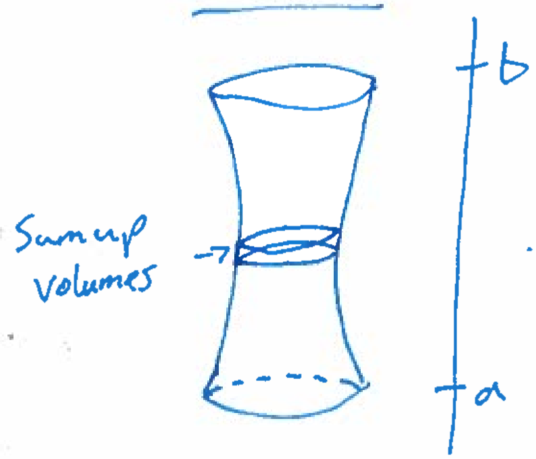
$$\Big\downarrow {\scriptstyle n \to \infty}$$

$$\int_a^b f(x)\,dx$$

$$\text{"}$$

Area

3-dim



Let $A(x_i)$ = Area of cross-section.

Then $\displaystyle\sum_{i=1}^{n} A(x_i)\cdot\Delta x$

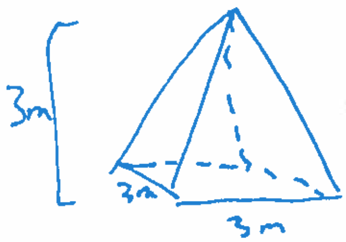
$$\Big\downarrow {\scriptstyle n \to \infty}$$

$$\int_a^b A(x)\,dx$$

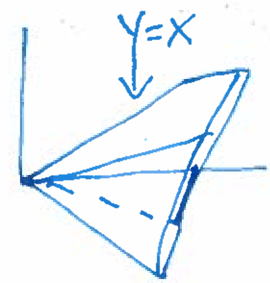
$$\text{"}$$

Volume

Ex 1: Pyramid 3m high has square base 3m×3m.



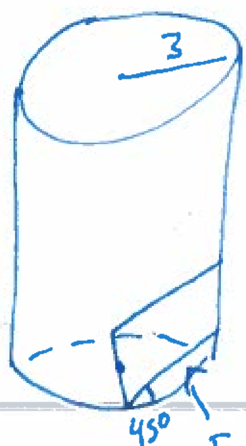
or

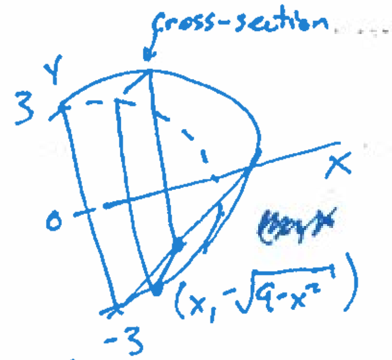


Area of a cross section is x^2.

So volume $= \int_0^3 x^2\,dx = \frac{x^3}{3}\Big|_0^3 = 9\,m^3$.

Ex 2:





Find this volume

Area of Cross-section $=$

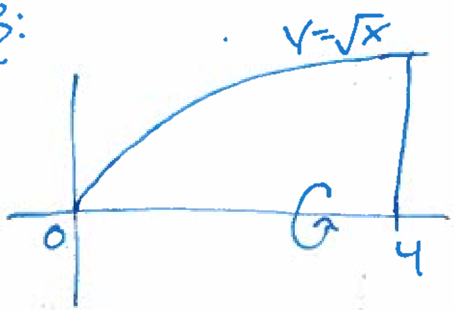
$$A(x) = x \cdot 2\sqrt{9-x^2}.$$

So Volume $= \int_0^3 x \cdot 2\sqrt{9-x^2}\,dx = 18$ by substitution method.

Solids of Revolution: (Disk and Washer Method)

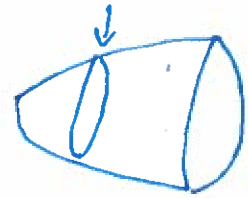
Area of Circle $= \pi r^2$.

Ex 3:



Rotate around $\rightarrow$ x-axis

cross section

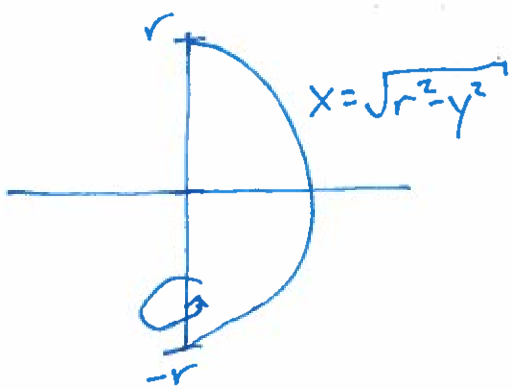


$$\text{Area} = A(x) = \pi(\sqrt{x})^2 = \pi x.$$

$$\text{Thus Volume} = \int_0^4 \pi x\,dx = \frac{\pi x^2}{2}\Big|_0^4 = 8\pi.$$

Ex 4: Volume of a sphere.

Circle centered at Origin $:= x^2 + y^2 = r^2$



$\longrightarrow$ rotate around Y-axis

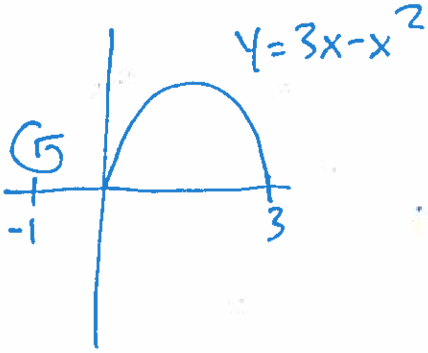
cross section

$$\text{Area} = \pi(\sqrt{r^2-y^2})^2 = \pi(r^2-y^2)$$

$$\text{Thus Volume} = \int_{-r}^{r} \pi(r^2-y^2)\,dy = \pi\left(r^2y - \frac{y^3}{3}\right)\Big|_{-r}^{r}$$

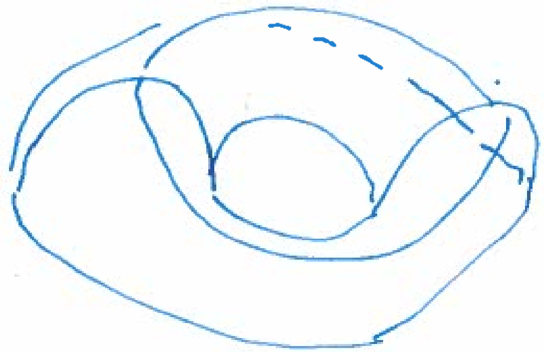
$$= \frac{4}{3}\pi r^3.$$

6.2: Volumes using Cylindrical Shells

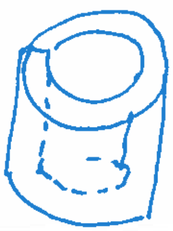


rotate around $x=-1$

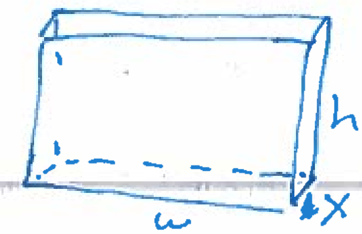
Bunt Cake



Cross sections are now cylindrical shells instead of washers.



unravel



Circumference $\longrightarrow$ width $= 2\pi\cdot(1+x)$

height $\longrightarrow$ height $= 3x-x^2$

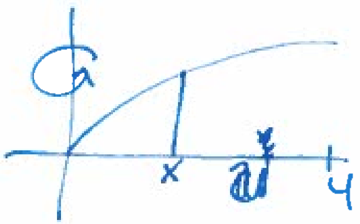
$\Delta r \longrightarrow \Delta x$

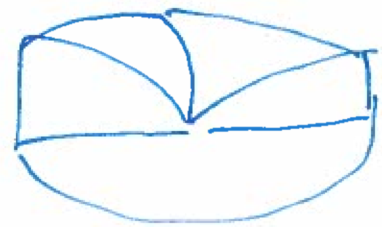
$$\text{Area} = 2\pi(1+x)(3x-x^2)\Delta x$$

$$\text{Volume} = \int_0^3 2\pi(1+x)(3x-x^2)\,dx = \frac{45\pi}{2}.$$

$$\text{Volume} = \int_a^b 2\pi\,(\text{shell radius})(\text{shell height})\,dx$$

Ex (1): $y=\sqrt{x}$



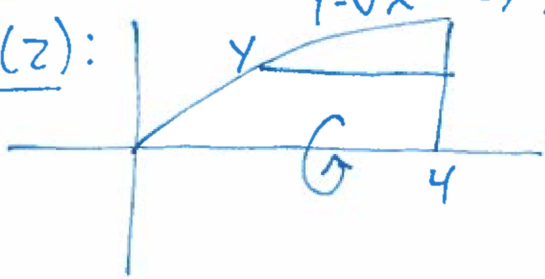


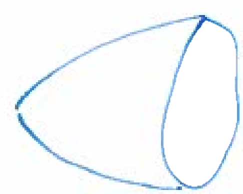
radius $= x$

height $= \sqrt{x}$

$$\text{Volume} = \int_0^4 2\pi \cdot x \cdot \sqrt{x}\, dx = 2\pi \left.\frac{x^{5/2}}{5/2}\right|_0^4 = \frac{128\pi}{5}.$$

Ex (2): $y=\sqrt{x} \Rightarrow x=y^2$





radius $= y$

height $= y^2$

$$\text{Volume} = \int_0^2 2\pi \cdot y \cdot (4-y^2)\, dy = \left.\pi\left(4y^2 - \frac{y^4}{2}\right)\right|_0^2 = 8\pi$$